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FIXED POINT THEOREMS AND THEIR APPLICATION – DISCRETE VOLTERRA OPERATORS

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Fixed Point Theorems and their Application – Discrete Volterra Operators

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Abstract The existence of solutions of nonlinear discrete Volterra equations is established. We define discrete Volterra operators on normed spaces of infinite sequences of finite-dimensional vectors, and present some of their basic properties (continuity, boundedness, and representation). The treatment relies upon the use of coordinate functions, and the existence results are obtained using fixed point theorems for discrete Volterra operators on infinite-dimensional spaces based on fixed point theorems of Schauder, Rothe, and Altman, and Banach's contraction mapping theorem, for *finite*-dimensional spaces.

The use of coordinate functions allows one to treat completely general discrete Volterra operators including those giving rise to Volterra equations of the form

$$\begin{aligned} x(n) = \quad f(n) \quad & + \sum_{\ell_1=0}^{n} g_1(n,\ell_1,x(\ell_1)) + \sum_{\ell_2=0}^{n}\sum_{\ell_1=0}^{n} g_2(n,\ell_1,\ell_2,x(\ell_1),x(\ell_2)) + \\ & + \sum_{\ell_3=0}^{n}\sum_{\ell_2=0}^{n}\sum_{\ell_1=0}^{n} g_3(n,\ell_1,\ell_2,\ell_3,x(\ell_1),x(\ell_2),x(\ell_3)) + \cdots \\ & + \sum_{\ell_k=0}^{n}\cdots\sum_{\ell_2=0}^{n}\sum_{\ell_1=0}^{n} g_k(n,\ell_1,\ell_2,\cdots,\ell_k,x(\ell_1),x(\ell_2),\cdots,x(\ell_k)) \\ & \qquad\qquad (n \geq 0). \end{aligned}$$

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1 Introduction

The existence of solutions of nonlinear discrete Volterra equations is established. We define discrete Volterra operators on normed spaces of infinite sequences of finite-dimensional vectors. and present some of their basic properties (continuity, boundedness, and representation). The treatment relies upon the use of coordinate functions, and the existence results are obtained using fixed point theorems for discrete Volterra operators on infinite-dimensional spaces based on fixed point theorems of Schauder, Rothe, and Altman, and Banach's contraction mapping theorem, for finite-dimensional spaces. Our main aim is to establish an existence theory for discrete Volterra equations (or implicit Volterra difference equations). Volterra equations are (see Corduneanu [9]) equations of non-anticipative type; the most familiar examples are perhaps Volterra integral equations of the second kind [5]. Since the classical work of Volterra [38], the theory of linear and nonlinear Volterra equations has played an important role in the application of mathematics in various disciplines. Discrete Volterra equations (DVEs) can be regarded as the discrete analogue of Volterra integral equations, and they arise directly, in modelling systems that are inherently digital such as digital filters and computer-controlled systems where the inputs and outputs are sampled discretely [15] and indirectly in the discretization of integral equations [1, 5, 8, 21, 25, 30].

An example of a discrete Volterra equation is

$$x(n) = f(n) + \sum_{j=0}^{n} g(n, j, x(j)), \quad n \geq 0 \tag{1}$$

where $\{f(n)\}_{n=0}^{\infty}$ is given, along with the form of $g(\cdot, \cdot, \cdot)$, and we seek $\{x(n)\}_{n=0}^{\infty}$. A more complicated example is

$$x(n) = f(n) + \sum_{j=0}^{n} g_1(n, j, x(j)) + \sum_{k=0}^{n} \sum_{j=0}^{n} g_2(n, j, k, x(j), x(k)), \quad n \geq 0. \tag{2}$$

This is one of a nested sequence of examples, of increasing complexity, the next in sequence being

$$\begin{aligned} x(n) &= f(n) + \sum_{j=0}^{n} g_1(n,j,x(j)) + \sum_{k=0}^{n}\sum_{j=0}^{n} g_2(n,j,k,x(j),x(k)) + \qquad (3)\\ &\qquad + \sum_{\ell=0}^{n}\sum_{k=0}^{n}\sum_{j=0}^{n} g_3(n,j,k,\ell,x(j),x(k),x(\ell)) \quad (n \geq 0). \end{aligned}$$

Discrete Volterra equations can be expressed in an abstract framework, on introducing Volterra operators on suitable spaces. (The coordinate functions employed by Yihong Song in his thesis [35] are natural but valuable tools.) The existence problem for solutions of discrete Volterra equations arises in the case that the equations are implicit. For linear implicit equations, the issues are readily solved; to treat nonlinear equations, we appeal to an appropriate fixed-point theorem in an abstract setting.

In section 3, we establish some fixed point theorems for discrete Volterra operators based on Schauder's fixed point theorem and the contraction mapping theorem. We also extend Rothe's and Altman's fixed point theorems to discrete Volterra operators. In section 4, as an application of fixed point theorems established in section 3, we investigate several types of discrete Volterra equations and prove that they have at least one solution under certain conditions.

For recent work related to discrete Volterra equations, the reader is referred in particular to the papers of Crisci and her co-workers [10]-[13], Elaydi and his co-workers [16]-[24] and Kolmanovskii and his co-workers [27]-[30], to mention a few; Agarwal and his co-workers have worked, in particular, on finite difference equations (with a bounded and with an unbounded lag). Much of the existing literature relates to *stability*, some relates to periodic solutions, which we address elsewhere. It may be remarked that the discussion of solutions with a bounded norm is intimately related to stability issues. (This paper is one of a series; in succeeding papers we consider the existence of periodic solutions, and perturbation theory and stability; see [6] and [36, 37] respectively.)

2 Basic theory for discrete Volterra operators

2.1 Preliminaries

In what follows results labelled Theorem will have a proof supplied but the proof of a result stated as a Proposition will be omitted if it is considered to be obvious or if a reference to the existing literature suffices.

We present some basic definitions.

Definition 1. *We denote by $\mathbb{Z}^+ = \{0, 1, \ldots, \}$ the set of all nonnegative integers, by $\mathbb{R}^d$ the d-dimensional real Euclidean space, by $\mathbb{C}^d$ the d-dimensional complex space, and*

$$\mathbb{E}^d \textit{ denotes either } \mathbb{R}^d \textit{ or } \mathbb{C}^d.$$

We need to generalize the familiar definitions of ℓ_p, the spaces of infinite sequences ($\alpha = \{\alpha_1, \alpha_2, \alpha_3, \cdots\}$) of scalars α_i, with finite p-norm $\|\alpha\|_p$. Introducing a linear space $\mathbb{X}$, the linear space of infinite sequences $\{\phi_1, \phi_2, \phi_3, \cdots\}$ with $\phi_i \in \mathbb{X}$ becomes a linear space in an obvious fashion. If $\mathbb{X}$ comes equipped with a norm $|\cdot|$ we can introduce the norm $\|\phi\|_p := \|\alpha(\phi)\|_p$ where $\alpha(\phi) = \{|\phi_1|, |\phi_2|, |\phi_3|, \cdots\cdots\}$. The case of particular interest to us is the case where $\mathbb{X}$ is a linear space of finite dimension $\dim(X) = d < \infty$. We now formalize this discussion through the medium of a definition.

Definition 2. *Let $(\mathbb{X}; |\cdot|)$ be a Banach space (also denoted $\mathbb{X}$). We denote by $\mathcal{S}(\mathbb{X})$ the set*

$$\mathcal{S} \equiv \mathcal{S}(\mathbb{X}) = \{\phi : \phi = \{\phi(n)\}_{n=0}^{\infty},\ \phi(n) \in \mathbb{X}\}, \tag{4}$$

which, with the obvious definition of addition and scalar multiplication, is a linear space of sequences of elements of $\mathbb{X}$. We employ the notation $\phi \equiv \{\phi(n)\}_{n=0}^{\infty}$, and $||\phi||_p = (\sum_{n=0}^{\infty} |\phi(n)|^p)^{1/p}$ $(1 \leq p < \infty)$ and $||\phi||_\infty = \sup_{n\geq 0} |\phi(n)|$, and we define

$$\mathcal{S}^p(\mathbb{X}) := \{\phi\ |\phi \in \mathcal{S}(\mathbb{X}) \quad \textit{with} \quad \|\phi\|_p < \infty\} \qquad (1 \leq p \leq \infty), \tag{5}$$

the Banach space consisting of elements of $\mathcal{S}(\mathbb{X})$, with the norm $\|\phi\|_p$. In each space we define the ball with radius r centered on the null sequence:

$$B^p(\mathbb{X}, r) := \{\phi\ |\phi \in \boldsymbol{\ell}^p,\ \|\phi\|_p \leq r\} \quad (1 \leq p \leq \infty). \tag{6}$$

The corresponding ball with radius r but centered on ψ *will be denoted* $B^p(\psi; r)$ *(*$\chi \in B^p(\psi; r)$ *if and only if* $\chi - \psi \in B^p(\mathbb{X}, r)$*).*

Consider the case where $d = \dim(\mathbb{X})$ is finite.

Definition 3. *Choose* $\mathbb{X} = \mathbb{E}^d$ *in Definition 2 to define*

$$\boldsymbol{\ell}^p \equiv \boldsymbol{\ell}^p(\mathbb{E}^d) = \mathcal{S}^p(\mathbb{E}^d). \tag{7}$$

Thus, $\boldsymbol{\ell}^p$ denotes the Banach space comprising sequences (of vectors) with finite norm $\|\phi\|_p$ where

$$\|\phi\|_p = (\sum_{n=0}^{\infty} |\phi(n)|^p)^{1/p} \quad (1 \le p < \infty) \text{ and } \|\phi\|_\infty = \sup_{n \ge 0} |\phi(n)|. \tag{8}$$

Consider sequences whose terms eventually vanish and sequences with a finite number of terms.

Definition 4. *We denote by* $\mathcal{S}_m \equiv \mathcal{S}_m(\mathbb{X})$ *(*$m \ge 0$*) the linear space of terminating sequences* $\{\phi(n)\}_{n=0}^m$ *(*$\phi(n) \in \mathbb{X}$*) and by* $\mathcal{S}_m^p \equiv \mathcal{S}_m^p(\mathbb{X})$ *the corresponding Banach space with norm* $|\cdot|_m^p$*:*

$$|\varphi|_m^p = (\sum_{n=0}^{m} |\phi(n)|^p)^{1/p} \; (1 \le p < \infty) \textit{ and } |\varphi|_m^\infty = \sup_{0 \le n \le m} |\phi(n)|. \tag{9}$$

The ball $B_m^p(\mathbb{X}, r)$ *is the obvious analogue of (6) and* $B_m^p(\psi; r)$ *is the analogue of* $B^p(\psi; r)$ *(*$\chi \in \mathcal{S}_m^p(\mathbb{X})$ *is in* $B_m^p(\psi; r)$ *if and only if* $\chi - \psi \in B_m^p(r)$*).*

Suppose $\phi = \{\phi(n)\}_{n=0}^\infty \in \mathcal{S}(\mathbb{X})$*. We call* $\{\phi(n)\}_{n=0}^m$ *the truncation of* $\{\phi(n)\}_{n=0}^\infty$ *and write*

$$\phi\rceil_m = \{\phi(n)\}_{n=0}^m \textit{ (denoting the 'projection' of } \phi \in \mathcal{S}(\mathbb{X}) \textit{ on } \mathcal{S}_m(\mathbb{X})\textit{).} \tag{10}$$

Similarly, for any subset $\mathcal{D} \subset \mathcal{S}(\mathbb{X})$*, define* $\mathcal{D}_i$ *as*

$$\mathcal{D}_i = \{\phi\rceil_i : \phi\rceil_i = \{\phi(n)\}_{n=0}^i \quad \textit{for all } \{\phi(n)\}_{n=0}^\infty \in \mathcal{D}\}$$

We call $\{\phi(0), \phi(1), \ldots, \phi(m), 0, 0, \ldots\}$ *the minimal extension of* $\{\phi(n)\}_{n=0}^m$*,*

Finally we consider terminating sequences with $\mathbb{X} = \mathbb{E}^d$. Clearly, $\mathcal{S}_m^p(\mathbb{E}^d)$ is a $(m+1) \times d$ dimensional Banach space and is isometric to the $(m+1) \times d$ dimensional subspace of $\boldsymbol{\ell}^p(\mathbb{E}^d)$ under the correspondence

$$\phi\rceil_m = \{\phi(n)\}_{n=0}^m \longleftrightarrow \phi = \{\phi(0), \phi(1), \ldots, \phi(m), 0, 0, \ldots\}.$$

We note that $B_m^p(\psi; r)$ is a closed, bounded, convex subset of $\mathcal{S}_m^p(\mathbb{X})$.

2.2 Discrete Volterra operators

We give some examples of Volterra operators acting on the space $\mathcal{S}(\mathbb{E}^d)$ defined in (4).

Example 1. *Given a mapping $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{E}^d \to \mathbb{E}^d$ and $f = \{f(n)\}_{n=0}^{\infty} \in \mathcal{S}(\mathbb{E}^d)$, define a corresponding operator V on $\mathcal{S}(\mathbb{E}^d)$ by the relation*

$$(V\phi)(n) = f(n) + \sum_{j=0}^{n} g(n, j, \phi(j)), \quad n \geq 0, \tag{11a}$$

for $\phi = \{\phi(n)\}_{n=0}^{\infty} \in \mathcal{S}(\mathbb{E}^d)$. The operator V in (11a) is a Volterra operator on $\mathcal{S}(\mathbb{E}^d)$; compare eqn. (1): writing (1) as $x(n) = (Vx)(n)$, x is a fixed point *of V. The theory is sometimes facilitated by the introduction of a scalar parameter λ as in $(V_\lambda \phi)(n) = f(n) + \lambda \sum_{j=0}^{n} g(n, j, \phi(j))$, $n \geq 0$.*

Special subcases of (11a) occur frequently. One common special form of (11a) is

$$(V\phi)(n) = f(n) + \sum_{j=0}^{n} K(n, j) g(j, \phi(j)), \quad n \geq 0, \tag{11b}$$

where $\{K(n, m)\}_{0 \leq m \leq n}$, $n \geq 0$, are $d \times d$ matrices, $g : \mathbb{Z}^+ \times \mathbb{E}^d \to \mathbb{E}^d$ is a mapping. In this case, the mapping $K : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{E}^d$ is called the kernel. The special form

$$(V\phi)(n) = f(n) + \sum_{j=0}^{n} g(n - j, \phi(j)), \quad n \geq 0, \tag{11c}$$

is called a nonlinear discrete Volterra operator of convolution type.

Remark 1. (a) The preceding examples arise, in particular, on applying appropriate quadrature rules $\displaystyle\int_0^{nh} \psi(t)\mathrm{d}t \approx h \sum_{j=0}^{n} W_{nj} \psi(jh)$ to the discretization of

$$\mathfrak{x}(t) = \mathfrak{f}(t) + \int_0^t \mathfrak{g}(t, s, \mathfrak{x}(s))\mathrm{d}s, \tag{12a}$$

(or to $\mathfrak{x}(t) = \mathfrak{f}(t) + \displaystyle\int_0^t \mathfrak{K}(t, s)\mathfrak{g}(s, x(s))\mathrm{d}s$, $\mathfrak{x}(t) = \mathfrak{f}(t) + \displaystyle\int_0^t \mathfrak{g}(t - s, \mathfrak{x}(s))\mathrm{d}s$, etc). If we apply suitable quadrature to

$$\mathfrak{x}(t) = \mathfrak{f}(t) + \int_0^t \mathfrak{g}_1(t, s, \mathfrak{x}(s))\mathrm{d}s + \int_0^t \int_0^t \mathfrak{g}_2(t, s, u, \mathfrak{x}(s), \mathfrak{x}(u))\mathrm{d}s\mathrm{d}u, \quad t \geq 0, \tag{12b}$$

we produce an example of (2) as the discretized version.

(b) Discrete Volterra equations also arise from the discretrization of Volterra integral equations with bounded memory, such as

$$\mathfrak{x}(t) = \mathfrak{f}(t) + \int_{t-\tau}^{t} \mathfrak{K}(t,s)\mathfrak{g}(s,\mathfrak{x}(s))\mathrm{d}s, \text{ where } \tau > 0, \tag{12c}$$

and, using (explicit or implicit) linear multistep formulae, from evolutionary ordinary differential equations (e.g., $\mathfrak{x}'(t) = \mathfrak{g}(t,\mathfrak{x}(t))$), delay differential equations (e.g., $\mathfrak{x}'(t) = \mathfrak{g}(t,\mathfrak{x}(t),\mathfrak{x}(t-1))$), and also, with finite-difference approximations, from certain partial differential equations.

We are interested in a formal definition of a discrete Volterra operator because of the association with discrete Volterra equations.

Definition 5. *An operator V from $\mathcal{D} \subseteq \mathcal{S}(\mathbb{X})$ into $\mathcal{R} \subseteq \mathcal{S}(\mathbb{X})$ will be called a* (discrete) Volterra operator *on $\mathcal{D}$ if there exists a family of mappings $v_i : \mathcal{D}_i \to \mathbb{X}$, $i \geq 0$, such that, for each $\phi = \{\phi(n)\}_{n=0}^{\infty} \in \mathcal{D}$,*

$$(V\phi)(n) = v_n(\phi(0), \phi(1), \ldots, \phi(n)), \quad n = 0, 1, \ldots. \tag{13}$$

The mapping v_n is called the n-th coordinate mapping *of V. $\mathcal{D}$ is the domain of V and $\mathcal{R}$ is the range of V.*

The coordinate mappings are very natural since V is usually defined explicitly by giving $(V\phi)(n)$. We note the following, in the light of Remark 1(b).

Definition 6. *If the coordinate functions (13) are of the form*

$$v_n(\phi(0), \phi(1), \ldots, \phi(n)) \equiv s_n(\phi(n-k_n), \phi(n-k_n+1), \ldots, \phi(n)),\ n = 0, 1, \ldots,$$

where, for some fixed k, $k_n \leq \min\{k, n\}$ for all n, then the operator V, and each coordinate function, can be said to have a bounded memory*. If the coordinate functions (13) are of the form $v_n(\phi(0), \phi(1), \ldots, \phi(n)) \equiv r_n(\phi(0), \phi(1), \ldots, \phi(n-1))$, $n = 0, 1, \ldots$, then the coordinate functions can be called* explicit.

It is obvious that the identity operator $I : (I\phi)(n) = \phi(n)$ $(n \geq 0)$ for all $\phi \in \mathcal{S}(\mathbb{X})$, is a Volterra operator. If V is a Volterra operator, and $f = \{f(n)\}_{n\geq 0} \in \mathcal{S}$ is fixed, then the operator Q, defined by $(Q\phi)(n) = f(n) \pm (V\phi)(n)$, $n \geq 0$, is also a Volterra operator. Let V and Q be Volterra operator. Then $V \pm Q$, VQ and QV are Volterra operators. In general, VG and GV are not Volterra operators if V is a Volterra operator but G is not.

A discrete Volterra operator is "causal" or "non-anticipative" and therefore provides an example of what is known as an (abstract) Volterra operator [9]. When $\mathcal{S}(\mathbb{X})$ is furnished with a norm, we can define continuity of an operator on $\mathcal{S}(\mathbb{X})$ in the obvious way. If the discrete Volterra operator V is continuous at each point of $S \subseteq \boldsymbol{\ell}^p(\mathbb{X})$ (where $1 \le p \le \infty$), we say that V is continuous on $S(\mathbb{X})$.

2.3 Linear and nonlinear discrete Volterra operators

The following result is clear.

Proposition 1. *Let V be a Volterra operator on $\mathcal{S}(\mathbb{X})$ and let $\{v_i\}_{i\ge 0}$ be its coordinate mappings. Then V is a linear operator if and only if each coordinate mapping v_i $(i \ge 0)$ is a linear mapping on $\mathcal{S}_i(\mathbb{X})$.*

In the case that $\mathbb{X}$ is finite dimentional and V is linear, we have a represention theorem.

Theorem 1. *Let $V : \mathcal{S}(\mathbb{E}^d) \to \mathcal{S}(\mathbb{E}^d)$ be a linear Volterra operator. Then there exists a family of $d \times d$ matrices $\{K(n,m)\}$, $0 \le m \le n$, $n \ge 0$, such that*

$$(V\phi)(n) = \sum_{j=0}^{n} K(n,j)\phi(j), \quad n \ge 0, \tag{14}$$

for any $\phi = \{\phi(n)\}_{n\ge 0} \in \mathcal{S}(\mathbb{E}^d)$.

Proof. Suppose $V : \mathcal{S}(\mathbb{E}^d) \to \mathcal{S}(\mathbb{E}^d)$ is a linear Volterra operator and v_i $(i \ge 0)$ are its coordinate mappings. Consider the structure of each $v_i : \mathcal{S}_i(\mathbb{E}^d) \to \mathbb{E}^d$ for fixed $i \ge 0$; for simplicity, consider v_1. Since any $(\phi(0), \phi(1)) \in \mathcal{S}_1(\mathbb{E}^d)$ can be written in the form $(\phi(0), \phi(1)) = (\phi(0), 0) + (0, \phi(1))$, it follows that

$$v_1((\phi(0), \phi(1))) = v_1((\phi(0), 0)) + v_1((0, \phi(1))).$$

On the other hand, the set $V = \{(x, 0) : x \in \mathbb{E}^d\}$ is isometric to $\mathbb{E}^d$. If we define a mapping $L : \mathbb{E}^d \to \mathbb{E}^d$ by the relation $Lx := v_1((x, 0))$, $x \in \mathbb{E}^d$, then L is a linear mapping on $\mathbb{E}^d$. From the representation theory of linear operators on the finite-dimensional space $\mathbb{E}^d$, it follows that there exists a $d \times d$ matrix, denoted by $K(1, 0)$ such that $v_1((\phi(0), 0)) = K(1, 0)\phi(0)$. Similarly, there exists a $d\times d$ matrix, denoted by $K(1,1)$ such that $v_1((0,\phi(1)) = K(1,1)\phi(1)$. Since the above argument can be applied to any v_i, $i \geq 0$, we obtain the stated theorem. □

Proposition 2. *Suppose that for $N \in \mathbb{Z}^+$ and for some fixed $f \in \mathcal{S}(\mathbb{X})$, the coordinate mappings v_i $(i \geq 0)$ of the operator V defined on $\mathcal{D}$ are*

$$v_n(\phi) = f(n) + \sum_{k=1}^{N} \Big\{ \sum_{j_M=0}^{n} \cdots \sum_{j_2=0}^{n} \sum_{j_1=0}^{n} g_k(n, j_1, j_2, \cdots, j_k, \phi(j_1), \phi(j_2), \cdots, \ \phi(j_M)) \Big\} \tag{15}$$

where the functions g_k assume values in $\mathbb{X}$. Then V is a (nonlinear, discrete) Volterra operator on $\mathcal{D}$.

2.4 Discrete Volterra operators on $\boldsymbol{\ell}^\infty(\mathbb{X})$

Let V be a discrete Volterra operator on $\mathcal{D} \subset \mathcal{S}(\mathbb{X})$ and $\phi \in \mathcal{D}$, with $(V\phi)(n) \in \mathbb{X}$. We may say that V is bounded at ϕ if there exists a constant $M \geq 0$ such that $|(V\phi)(n)| < M$ for all $n \geq 0$, using the norm of $\mathbb{X}$. If V is bounded at each point of $\mathcal{D}$, then we actually have an operator V from $\mathcal{D}$ to $\boldsymbol{\ell}^\infty(\mathbb{X})$. Restricting the domain of V to a subset $\mathcal{D} \subset \boldsymbol{\ell}^\infty(\mathbb{X})$, we thus define a discrete Volterra operator $V : \mathcal{D} \to \boldsymbol{\ell}^\infty(\mathbb{X})$ (we use the same notation for the restriction as for the original operator). Similarly, for any subset $\mathcal{D} \subset \boldsymbol{\ell}^p(\mathbb{X})$ $(1 \leq p < \infty)$, we can define a discrete Volterra $V : \mathcal{D} \to \boldsymbol{\ell}^p(\mathbb{X})$.

We now consider discrete Volterra operators on $\boldsymbol{\ell}^\infty(\mathbb{X})$.

Definition 7. *Let $\phi \in \boldsymbol{\ell}^\infty(\mathbb{X})$. The coordinate functions $\{v_i\}_{i\geq 0}$ in (13) are called* equi-continuous *at $\phi]_i \in \mathcal{S}_i^\infty(\mathbb{X})$, $i = 0, 1, \ldots$, if, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|v_i(\phi]_i) - v_i(\psi_i)| \leq \varepsilon, \quad i = 0, 1, \ldots,$$

for any $\psi_i \in \mathcal{S}_i^\infty(\mathbb{X})$ satisfying $|\phi]_i - \psi_i|_i^\infty \leq \delta$ uniformly for all $i = 0, 1, \ldots$, where $v_i(\psi) = v_i(\psi(0), \psi(1), \ldots, \psi(i))$ for any $\psi = \{\psi(n)\}_{0\leq n\leq i} \in \mathcal{S}_i^\infty(\mathbb{X})$.

The proof of the next Proposition is straightforward if we note the relation

$$\|V\phi\|_\infty = \sup_{n\geq 0}\{|v_n(\phi(0), \ldots, \phi(n))|\} \quad \text{for} \quad \phi = \{\phi(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{X})$$

in terms of the coordinate functions.

Proposition 3. *Let V be a Volterra operator on $\boldsymbol{\ell}^\infty(\mathbb{X})$ and $\{v_i\}_{i\geq 0}$ be its coordinate mappings. Then V is continuous at $\phi = \{\phi(n)\}_{n\geq 0}$ if and only if v_i $(i \geq 0)$ is equi-continuous at $\phi]_i = \{\phi(n)\}_{0\leq n\leq i} \in \mathcal{S}_i^\infty(\mathbb{X})$*

Remark 2. Notice that if V is a continuous Volterra operator on $\boldsymbol{\ell}^\infty(\mathbb{X})$, then its coordinate mappings $v_i : \mathcal{S}_i^\infty(\mathbb{X}) \to \mathbb{X}$ $(i \geq 0)$ are continuous. Generally, the converse conclusion does not hold. Even the continuity of all the coordinate mappings $v_i : \mathcal{S}_i^\infty(\mathbb{X}) \to \mathbb{X}$ $(i \geq 0)$ of V does not guarantee that V is continuous; what is required is the equi-continuity of the family of coordinate functions. However, our main results for Volterra operators in this paper only require that each coordinate mapping of the Volterra operator is continuous.

Proposition 4. *Let V be a Volterra operator on $\boldsymbol{\ell}^\infty(\mathbb{X})$, and let $\{v_i\}_{i\geq 0}$ be its coordinate mappings. Then V is bounded on $\boldsymbol{\ell}^\infty(\mathbb{X})$, namely, there is a number $M > 0$ such that $|(V\phi)(n)| \leq M$ for all $n \geq 0$ and $\phi \in \boldsymbol{\ell}^\infty(\mathbb{X})$ if and only if each $v_i : \mathcal{S}_i^\infty(\mathbb{X}) \to \mathbb{X}$, $i \geq 0$ is bounded uniformly by a constant $M > 0$, that is $|v_i(\phi)| \leq M$ for all $\phi \in \mathcal{S}_i^\infty(\mathbb{X})$ and $i \geq 0$.*

3 Fixed point theory for discrete Volterra operators

3.1 Theorems derived from Schauder's fixed point theory

There are a number of topological fixed point theorems which go back to Brouwer [7] and Schauder [33], and others. The classical statement of Schauder's theorem relates to the existence of a fixed point of a compact operator mapping a closed and bounded convex subset of a Banach space into itself. However, because the Banach space $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$ is constructed on $\mathcal{S}(\mathbb{E}^{d^d})$ and $\mathbb{E}^d$ is a finite dimensional space, we only require fixed point theorems for finite dimensional spaces.

Proposition 5. *(a) Any continuous mapping of a convex subset Ω of $\mathbb{E}^d$ into a bounded closed set inside Ω has one fixed point.*

(b) Any continuous mapping of $\mathbb{E}^d$ into a bounded subset of $\mathbb{E}^d$ has one fixed point.

Proof. See [34, Theorem 4.15].

We are now in a position to prove the main result in this section.

Theorem 2. *Let $V : \boldsymbol{\ell}^\infty(\mathbb{E}^d) \to \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ be a Volterra operator that is bounded (namely there is a number $M > 0$ such that $\|V\phi\|_\infty \leq M$ for all $\phi \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$). If all the coordinate mappings of V are continuous, then V has at least one fixed point in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$.*

Remark 3. If $V : \boldsymbol{\ell}^\infty(\mathbb{E}^d) \to \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ is compact and continuous, then V has a fixed point by Schauder's fixed point theorem (see, e.g., [34], Corollary 4.1.4). Unfortunately, establishing compactness of an operator on $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$ is usually not straightforward.

Remark 4. Obviously, if V is continuous on $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$, then its all coordinate mappings are continuous. The condition that V is of Volterra type is required in Theorem 2; generally, Theorem 2 is not valid if we replace the Volterra operator by a continuous operator on $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$. The same comment applies to all of our fixed point theorems for discrete Volterra operators based on Schauder's fixed point theorem.

Proof. Let v_i $(i \geq 0)$ be the coordinate mappings of V, so that $v_i : \mathcal{S}_i^\infty(\mathbb{E}^d) \to \mathbb{E}^d$ $(i \geq 0)$ is a continuous mapping. For each $m \geq 0$, we define an operator V_m on $\mathcal{S}_m^\infty(\mathbb{E}^d)$ by the relations

$$V_m(\phi)(n) = v_n(\phi(0), \phi(1), \ldots, \phi(n)) \quad (0 \leq n \leq m) \tag{16}$$

for each $\phi = \{\phi(n)\}_{n=0}^m \in \mathcal{S}_m^\infty(\mathbb{E}^d)$. It is obvious that V_m is continuous and bounded on $\mathcal{S}_m^\infty(\mathbb{E}^d)$. By the assumption of Theorem 2 there is a $M > 0$ such that

$$|V_m(\phi)| \leq M \quad \text{for all } \phi = \{\phi(n)\}_{0 \leq n \leq m} \in \mathcal{S}_m^\infty(\mathbb{E}^d),\ m = 0, 1, \ldots. \tag{17}$$

It follows from Proposition 5 that there exists at least one $\phi_m = \{\phi_m(n)\}_{n=0}^m$ such that (for $m = 0, 1, \ldots$)

$$\phi_m(n) = (V_m\phi_m)(n) = v_n(\phi_m(0), \phi_m(1), \ldots, \phi_m(n)),\ 0 \leq n \leq m. \tag{18}$$

Since $|\phi_m(n)| = |(V_m\phi_m)(n)| \leq M$ for all $m \geq 0$ and $0 \leq n \leq m$, it follows that the set $\{\phi_m(n)\}_{m \geq n}$ is a bounded set in $\mathbb{E}^d$ for fixed $n \geq 0$.

Let us first consider the set $\{\phi_m(0)\}_{m\geq 0}$. We note that each $\phi_m(0), m \geq 0$ satisfies (18) for $n = 0$, that is

$$\phi_m(0) = v_0(\phi_m(0)) \quad \text{for all} \quad m \geq 0. \tag{19}$$

Since $|\phi_m(0)| \leq M$, $m \geq 0$, it follows that the set $\{\phi_m(0)\}_{m\geq 0}$ is a bounded subset in $\mathbb{E}^d$. Thus it has a subsequence $\{\phi_{m_{0k}}(0)\}$, $0 < m_{00} < m_{01} < \ldots < m_{0k} < \ldots$, where $m_{0k} \to \infty$ as $k \to \infty$, which convergence to a point on $\mathbb{E}^d$, say $\phi(0)$. Since v_0 is continuous on $\mathbb{E}^d$ and $\{\phi_{m_{0k}}(0)\}$ satisfy (19), one gets, by setting $m = m_{0k}$ in (19) and letting $k \to \infty$,

$$\phi(0) = v_0(\phi(0)).$$

Since $m_{00} > 0$, $\phi_{m_{0k}}(1)$ satisfies (18) for $n = 1$, that is

$$\phi_{m_{0k}}(1) = v_1(\phi_{m_{0k}}(0), \phi_{m_{0k}}(1)), \quad k = 0, 1, \ldots. \tag{20}$$

¿From $|\phi_{m_{0k}}(1)| \leq M$, it follows that $\{\phi_{m_{0k}}(1)\}_{k\geq 0}$ is a bounded subset in $\mathbb{E}^d$, and thus has a subsequence $\{\phi_{m_{1k}}(1)\}$, $1 < m_{10} < m_{11} < \ldots < m_{1k} < \ldots$,$m_{1k} \to \infty$ as $k \to \infty$, which convergence to a point on $\mathbb{E}^d$, say $\phi(1)$. We note that $\{\phi_{m_{1k}}(0)\}$ is a subsequence of $\{\phi_{m_{0k}}(0)\}$. Then $\phi_{m_{1k}}(0) \to \phi(0)$ as $k \to \infty$. Since each $\phi_{m_{1k}}(1)$ satisfies (20), that is

$$\phi_{m_{1k}}(1) = v_1(\phi_{m_{1k}(0)}, \phi_{m_{1k}}(1)),$$

$k = 0, 1, \ldots$, and v_1 is also continuous, let $k \to \infty$ in the above equation, one gets

$$\phi(1) = v_1(\phi(0), \phi(1)).$$

In this way, we can show for each $q > 1$, inductively, that there exists a subsequence $\{\phi_{m_{qk}}(q)\}$, $q < m_{q0} < m_{q1} < \ldots < m_{qk} < \ldots$, $m_{qk} \to \infty$ as $k \to \infty$, of $\{\phi_{m_{(q-1)k}}(q-1)\}$ such that $\lim_{k\to\infty} \phi_{m_{qk}}(p) = \phi(p)$, $0 \leq p \leq q$, and

$$\phi_{m_{qk}}(q) = v_q(\phi_{m_{qk}}(0), \phi_{m_{qk}}(1), \ldots, \phi_{m_{qk}}(q)), \quad k \geq 0.$$

Let $k \to \infty$ in the above equation, we obtain

$$\phi(q) = v_q(\phi(0), \phi(1), \ldots, \phi(q)).$$

Thus, we obtain a sequence $\{\phi(n)\}_{n\geq 0} = \phi$ which satisfy equations

$$\phi(n) = v_n(\phi(0), \phi(1), \ldots, \phi(n)) \quad (n \geq 0)$$

with $\|\phi\|_\infty \leq M$, which implies that $\phi = \{\phi(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ is a fixed point of V. The proof is completed. $\square$

Corollary 1. *Suppose $V : B^\infty(\mathbb{E}^d, r) \to B^\infty(\mathbb{E}^d, r)$ is a continuous discrete Volterra operator. Then V has at least one fixed point in $B^\infty(\mathbb{E}^d, r)$.*

The condition that V is continuous on $B^\infty(\mathbb{E}^d, r)$ can be replaced by the condition that the coordinate mappings v_i of V are continuous on $B_i^\infty(r)$ for all $i \geq 0$.

Proof. It is obvious that if $V : B^\infty(\mathbb{E}^d, r) \to B^\infty(\mathbb{E}^d, r)$, then V_m maps $B_m^\infty(\mathbb{E}^d, r)$ to $B_m^\infty(\mathbb{E}^d, r)$, where V_m is the restriction of V on $\mathcal{S}_m^\infty(\mathbb{E}^d)$ defined by (16). Since $\mathcal{S}_m^\infty(\mathbb{E}^d)$ is finite dimensional space and V_m is continuous on $B_m^\infty(\mathbb{E}^d, r)$ by the assumption of Corollary 1, V_m has at least one fixed point $\phi_m \in B_m^\infty(\mathbb{E}^d, r)$ by Proposition 5 and $|\phi_m|_m^\infty \leq r$ for each $m \geq 0$. Using the same technique as that in the proof of Theorem 2, one can readily prove that V has at least one fixed point $\phi \in B^\infty(r)$. $\square$

Theorem 3. *Suppose $V : \boldsymbol{\ell}^\infty(\mathbb{E}^d) \to \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ is a Volterra operator and the operators V_m $(m \geq 0)$ (the restrictions of V on $\mathcal{S}_m^\infty(\mathbb{E}^d)$ defined by (16)) are continuous. Then V has at least one fixed point either in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$ if and only if each V_m $(m \geq 0)$ has at least one fixed point ϕ_m in $\mathcal{S}_m^\infty(\mathbb{E}^d)$, and there is a number $M > 0$ such that $|\phi_m|_m^\infty \leq M$ for all $m \geq 0$.*

The condition that V_m $(m \geq 0)$ are continuous is equivalent to the condition that that all coordinate mappings of V are continuous.

Proof. Suppose V has one fixed point $\phi = \{\phi(n)\}_{n\geq 0}$ in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$. Define $M = \|\phi\|_\infty < \infty$. Let $\phi\rceil_m = \{\phi(n)\}_{0\leq n\leq m}$, $m \geq 0$, denote the restriction of ϕ into $\mathcal{S}_m^\infty(\mathbb{E}^d)$. Since ϕ is a fixed point of V, it follows that $\phi\rceil_m$ is a fixed point of V_m and $|\phi\rceil_m|_m^\infty \leq M$ for all $m \geq 0$. Conversely, suppose each V_m $(m \geq 0)$ has one fixed point ϕ_m in $\mathcal{S}_m^\infty(\mathbb{E}^d)$ and there is a number $M > 0$ such that $|\phi_m|_m^\infty \leq M$ for all $m \geq 0$. Using the same technique as employed in the proof of Theorem 2, one can readily prove that V has at least one fixed point $\phi \in B^\infty(\mathbb{E}^d, r)$. $\square$

Since Schauder's fixed point theorem does not deal with the uniqueness of fixed points, Theorem 2 and Theorem 3 do not guarantee the uniqueness of fixed points. The following Corollary shows that discrete Volterra operators could have a unique fixed point under certain conditions.

Corollary 2. *Suppose $V : \boldsymbol{\ell}^\infty(\mathbb{E}^d) \to \boldsymbol{\ell}^\infty(\mathbb{E}^d)$, where V is a Volterra operator. Let V_m be the restriction of V on $\mathcal{S}_m^\infty(\mathbb{E}^d)$ defined by (16). If each V_m is continuous and has a unique fixed point $\phi_m \in \mathcal{S}_m^\infty(\mathbb{E}^d)$, and there is a number $M > 0$ such that $|\phi_m|_m^\infty \leq M$for all $m \geq 0$, then V has a* unique *fixed point in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$.*

Proof. Under the present assumptions, it follows from Theorem 3 that V has at least one fixed point in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$. Since the restriction of the fixed point of V on $\mathcal{S}_m^\infty(\mathbb{E}^d)$, $m \geq 0$, is also a fixed point of V_m, V cannot have two different fixed point in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$. $\square$

Since $\boldsymbol{\ell}^p(\mathbb{E}^d) \subset \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ $(1 \leq p < \infty)$, the conclusions of Theorem 2—3 and Corollary 1—2 hold for a discrete Volterra Operator defined on $\boldsymbol{\ell}^p(\mathbb{E}^d)$. For example, we give the following result without proof.

Theorem 4. *Let $V : \boldsymbol{\ell}^p(\mathbb{E}^d) \to \boldsymbol{\ell}^p(\mathbb{E}^d)$ $(1 \leq p < \infty)$ be a discrete Volterra operator that is bounded (namely there is a number $M > 0$ such that $\|V\phi\|_p \leq M$ for all $\phi \in \boldsymbol{\ell}^p(\mathbb{E}^d)$). If all the coordinate mappings of V are continuous, then V has at least one fixed point in $\boldsymbol{\ell}^p(\mathbb{E}^d)$.*

3.2 Theorems of Altman type for discrete Volterra operators

We commence with known results and then refine them.

3.2.1 Rothe's and Altman's theorems in a Banach space

In some applications it appears easy to prove that a compact mapping has at least one fixed point if it satisfies certain conditions. Such conditions were given by Rothe [32], Altman [3, 2, 4] and others.

Definition 8. *Let $\mathbb{Y}$ be a Banach space with norm $\|\cdot\|$ and $B_r(\mathbb{Y})$ be the closed ball of radius $r > 0$ in $\mathbb{Y}$, that is*

$$B_r(\mathbb{Y}) = \{x \,|x \in \mathbb{Y} \quad \textit{and} \quad \|x\| \leq r\};$$

then the radial retraction R of $\mathbb{Y}$ onto $B_r(\mathbb{Y})$ is defined by

$$Rx = x \textit{ if } x \in B_r(\mathbb{Y}) \;, \qquad Rx = rx/\|x\| \textit{ if } x \notin B_r(\mathbb{Y}). \tag{21}$$

Definition 9. *Denote $\partial B_r(\mathbb{Y}) = \{x \,|x \in \mathbb{Y}, \|x\| = r\}$ the boundary of $B_r(\mathbb{Y})$. Given an operator F acting on a Banach space $(\mathbb{Y}, \|\cdot\|)$ the conditions on F due, respectively, to Rothe, Altman, and Petryshyn read:*

Condition R (Rothe) $\qquad \|F(x)\| \leq \|x\| \textit{ for } x \in \partial B_r(\mathbb{Y});$ (22a)

Condition A (Altman) $\qquad \|x - F(x)\|^2 \geq \|F(x)\|^2 - \|x\|^2 \textit{ for } x \in \partial B_r(\mathbb{Y});$ (22b)

Condition P (Petryshyn) $\qquad \|x - F(x)\| \geq \|F(x)\| \textit{ for } x \in \partial B_r(\mathbb{Y}).$ (22c)

Proposition 6. *(i) If Condition (R) holds, then Condition (A) holds; (ii) If Condition (P) holds, then Condition (A) holds.*

Proof: See [26, p. 166].

Proposition 7. *If $F : B_r(\mathbb{Y}) \to \mathbb{Y}$ is compact and satisfies one of the conditions (R), (A) and (P), then F has one fixed point in $B_r(\mathbb{Y})$.*

Proof: See Rothe [32], Altman [3], Petryshyn [31].

We seek to prove a modification of Proposition 7 in the case that $\mathbb{Y}$ is a finite dimensional Banach space. We first point out that in this case the conclusion of Proposition 7 holds without the compactness condition on F. To prove the later claim, we need the following results.

Proposition 8. *Let the radial retraction R onto V is defined by (21). Then (i) R is a continuous mapping of $\mathbb{Y}$ onto $B_r(\mathbb{Y})$; (ii) if $Rx \in \mathrm{Int}(B_r(\mathbb{Y}))$ then $Rx = x$; (iii) if $x \notin B_r(\mathbb{Y})$ then $Rx \in \partial B_r(\mathbb{Y})$, where $\mathrm{Int}(B_r(\mathbb{Y})) = \{x \mid x \in \mathbb{Y} \quad and \quad \|x\| < r\}$.*

Proof: See [34].

3.2.2 Theorems of Altman type

We are now in a position to state and prove the required modification of Proposition 7, under the assumption that $\mathbb{Y}$ is finite dimensional, in particular if $\mathbb{Y} = \mathbb{Y}_d$, d-dimensional Banach space.

Theorem 5. *Let $\mathbb{Y}_d$ be a d-dimensional Banach space with the norm $\| \cdot \|$ and $B_r(\mathbb{Y}_d)$ the ball of radius $r > 0$ in $\mathbb{Y}_d$. Let $F : B_r(\mathbb{Y}_d) \to \mathbb{Y}_d$ and satisfy one of conditions (R), (A) and (P). Then F has one fixed point in $B_r(\mathbb{Y}_d)$.*

Proof. By Proposition 6, we only need to prove the Theorem 5 under the condition (A). Let R be the retraction mapping on $B_r(\mathbb{Y}_d)$ defined by (21). We define a mapping on $\mathbb{Y}_d$ with range in $B_r(\mathbb{Y}_d)$ by the relation $\gamma(x) = RF(x)$. It is clear that γ is continuous and maps $B_r(\mathbb{Y}_d)$ into $B_r(\mathbb{Y}_d)$. Since $\mathbb{Y}_d$ is a finite dimensional Banach space, there exists at least one fixed point in $B_r(\mathbb{Y}_d)$. Let this fixed point be x_0. In this case, we will show that x_0 is a fixed point of F.

Suppose (*Case 1:*) that $\|x_0\| < r$ or $x_0 \in \mathrm{Int}(B_r(\mathbb{Y}_d))$. From $x_0 = \gamma(x_0) = RF(x_0) \in \mathrm{Int}(B_r(\mathbb{Y}_d))$, it follows that $x_0 = F(x_0)$ by Proposition

8 (ii). Thus x_0 is a fixed point of F. Suppose, alternatively, (*Case 2:*) that $\|x_0\| = r$ or $x_0 \in \partial B_r(\mathbb{Y}_d)$. If $\|F(x_0)\| < r$, then it follows from Proposition 8 (ii) that $x_0 = \gamma(x_0) = RF(x_0) = F(x_0) \in \mathrm{Int}(B_r(\mathbb{Y}_d))$. This is a contradiction. Thus, $\|F(x_0)\| \geq r$. By the definition of R, we have $x_0 = \gamma(x_0) = RF(x_0) = rF(x_0)/\|F(x_0)\|$ if $\|F(x_0)\| \geq r$. Let $a = \|F(x_0)\|/r$, then, we can write $F(x_0) = a \times x_0$. It is clear that $a \geq 1$. Now we show that we have in fact $a = 1$. Indeed, since F satisfies condition (A), we have $\|x_0 - F(x_0)\|^2 = (1-a)^2r^2$ and $\|F(x_0)\|^2 - \|x_0\|^2 = (a^2-1)r^2$. It follows from these two equalities that condition (A) is not possible unless $a = 1$. Thus, x_0 is a fixed point of F. The Theorem is thus established. $\square$

By virtue of Theorem 5, we can amend Proposition 7 to apply to discrete Volterra operators on $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$:

Theorem 6. *Let $V : B^\infty(\mathbb{E}^d, r) \to \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ be a continuous Volterra operator such that*

$$\|T(\phi)\|_\infty \leq \|\phi\|_\infty \quad \text{for all} \quad \phi \in \partial B^\infty(\mathbb{E}^d, r). \tag{23}$$

Then V has at least one fixed point in $B^\infty(\mathbb{E}^d, r)$.

The condition that V is continuous can be replaced by the condition that its coordinate mappings $v_i : B_i^\infty(r) \to \mathbb{E}^d$ are continuous for all $i \geq 0$.

Proof. Let V_m $(m \geq 0)$ be the restriction of V on $\mathcal{S}_m^\infty(\mathbb{E}^d)$ defined by (16), $B_m^\infty(\mathbb{E}^d, r) = \{\phi : |\phi|_m^\infty \leq r\}$ is the ball of radius $r > 0$ in $\mathcal{S}_m^\infty(\mathbb{E}^d)$. It is obvious that $V_m : B_m^\infty(\mathbb{E}^d, r) \to \mathcal{S}_m^\infty(\mathbb{E}^d)$ by the assumptions of Theorem 6. Let $\phi = \{\phi(n)\}_{n=0}^m \in \partial B_m^\infty(\mathbb{E}^d, r)$, namely $|\phi|_m^\infty = r$. Define the minimal extension $\psi = \{\psi(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ by the relation

$$\psi(n) = \phi(n) \text{ if } 0 \leq n \leq m, \quad \psi(n) = 0 \text{ if } n > m.$$

It is obvious that $\|\psi\|_\infty = r$, namely $\psi \in \partial B^\infty(\mathbb{E}^d, r)$. By the condition (23), we have

$$\begin{aligned} |V_m(\phi)|_m^\infty &= \sup_{0\leq n\leq m} \{|(V_m\phi)(n)|\} \\ &= \sup_{0\leq n\leq m} \{|v_n(\phi(0), \ldots, \phi(n))\} \\ &\leq \sup_{n\geq 0}\{|v_n(\psi(0), \psi(1), \ldots, \psi(n))\} \leq \|\psi\|_\infty = r. \end{aligned}$$

Thus, the operator V_m $(m \geq 0)$ on $\mathcal{S}_m^\infty(\mathbb{E}^d)$ satisfy condition (R). By Theorem 5, V_m has at least one fixed point $\phi_m \in B_m^\infty(\mathbb{E}^d, r)$, namely, $|\phi_m|_m^\infty \leq r$ for $m \geq 0$. It follows from Theorem 3 that V has at least one fixed point in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$. The proof is completed. $\square$

Theorem 7. *Let $V : B^\infty(\mathbb{E}^d, r) \to \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ be a continuous discrete Volterra operator and v_i $(i \geq 0)$ be its coordinate mappings such that one of conditions*

$$|\phi(n) - v_n(\phi(0), \dots, \phi(n))|^2 \geq |v_n(\phi(0), \dots, \phi(n))|^2 - |\phi(n)|^2 \quad (n \geq 0) \tag{24}$$

and

$$|\phi(n) - v_n(\phi(0), \dots, \phi(n))| \geq |v_n(\phi(0), \dots, \phi(n))| \quad (n \geq 0) \tag{25}$$

holds for any $\phi = \{\phi(n)\}_{n \geq 0} \in \partial B^\infty(\mathbb{E}^d, r)$. Then V has at least one fixed point in $B^\infty(\mathbb{E}^d, r)$.

The condition that V is continuous can be replaced by the condition that its coordinate mappings $v_i : B_i^\infty(\mathbb{E}, r) \to \mathbb{E}^d$ are continuous for all $i \geq 0$.

Proof. We first note that condition (25) implies condition (24); we thus have to prove Theorem 7 under condition (24). Let V_m $(m \geq 0)$ be the restriction of V on $\mathcal{S}_m^\infty(\mathbb{E}^d)$ defined by (16) and $B_m^\infty(\mathbb{E}^d, r)$ be the ball of radius $r > 0$ in $\mathcal{S}_m^\infty(\mathbb{E}^d)$. Then (24) implies

$$(|\phi - V_m\phi|_m^\infty)^2 \geq (|V_m\phi|_m^\infty)^2 - (|\phi|_m^\infty)^2 \quad \text{for each} \quad m \geq 0,$$

where $\phi \in \partial B_m^\infty(\mathbb{E}^d, r)$. It follows from Theorem 5 that V_m has at least one fixed point $\phi_m \in B_m^\infty(\mathbb{E}^d, r)$, or $|\phi_m|_m^\infty \leq r$ for each $m \geq 0$. Therefore, V has at least one fixed point in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$ by Theorem 3. The proof is completed. □

3.3 Uniqueness of fixed points via the contraction mapping theorem

Since Schauder's fixed point theorems are not concerned with the uniqueness of fixed points, the fixed point theorems for discrete Volterra operators established in the former two sections do not guarantee the uniqueness of fixed points. But a unique fixed point is often required in applications. We give a result (Theorem 9) based on the *Contraction Mapping Theorem* (Theorem 8), which guarantees the uniqueness of fixed points.

We recall the definition of a contraction mapping and the *Contraction Mapping Theorem.*

Definition 10. *(See e.g., [34], p.2.) Let T be a mapping of a metric space $(\mathbb{Y}, \varrho)$ into $(\mathbb{Y}, \varrho)$. We say that T is a* contraction mapping *if there exists a number τ such that $0 < \tau < 1$ and*

$$\varrho(Tx, Ty) \leq \tau\varrho(x, y) \quad (\forall x, y \in \mathbb{Y}).$$

The following result is called the *Contraction Mapping Theorem.*

Theorem 8. *(See, e.g., [34], p.2.) Any contraction mapping of a complete non-empty metric space* $\mathbb{Y}$ *into* $\mathbb{Y}$ *has a unique fixed point in* $\mathbb{Y}$.

Let us first deal with a proposition about the contraction of discrete Volterra operators.

Proposition 9. *Let* $V : \boldsymbol{\ell}^\infty(\mathbb{X}) \to \boldsymbol{\ell}^\infty(\mathbb{X})$ *be a Volterra operator and* v_i $(i \geq 0)$ *be its coordinate mappings. If* $v_i : \mathcal{S}_i^\infty(\mathbb{X}) \to \mathbb{X}$ $(i \geq 0)$ *satisfy the following condition (H0):*

(**H0**) *there exist numbers* ρ_i, $0 < \rho_i < 1$ $(i \geq 0)$ *such that* $\sup_{i\geq 0} \rho_i < 1$ *and*

$$|v_i(\phi_i) - v_i(\psi_i)| \leq \rho_i |\phi_i - \psi_i|_i^\infty \tag{26}$$

for all $\phi_i = \{\phi(n)\}_{0\leq n\leq i}$, $\psi_i = \{\psi(n)\}_{0\leq n\leq i}$ *in* $\mathcal{S}_i^\infty(\mathbb{X})$,

then V *is a contraction operator, namely there exists a number* ρ *such that* $0 < \rho < 1$ *and*

$$\|(V\phi) - (V\psi)\|_\infty \leq \rho \|\phi - \psi\|_\infty \quad \forall \phi, \psi \in \boldsymbol{\ell}^\infty(\mathbb{X}). \tag{27}$$

Proof. The proof is obvious and is omitted. □

Theorem 9. *Let* $V : \boldsymbol{\ell}^\infty(\mathbb{X}) \to \boldsymbol{\ell}^\infty(\mathbb{X})$ *be a Volterra operator and* v_i $(i \geq 0)$ *be its coordinate mappings. If* $v_i : \mathcal{S}_i^\infty(\mathbb{X}) \to \mathbb{X}$ $(i \geq 0)$ *satisfy the condition (H0), then* V *has a unique fixed point in* $\boldsymbol{\ell}^\infty(\mathbb{X})$.

Proof. By Proposition 9, the proof is obvious by using Theorem 8, since V is a contraction mapping onto $\boldsymbol{\ell}^\infty(\mathbb{X})$. □

It is obvious Theorem 9 holds for a discrete Volterra operator defined on $B^\infty(\mathbb{X}, r)$.

Corollary 3. *Let* $V : B^\infty(\mathbb{X}, r) \to B^\infty(\mathbb{X}, r)$ *be a discrete Volterra operator and* v_i $(i \geq 0)$ *be its coordinate mappings. If* $v_i : B_i^\infty(\mathbb{X}, r) \to B_0^\infty(\mathbb{X}, r)$ *satisfy the condition (H0) for* $(i \geq 0)$, *then* V *has a unique fixed point in* $B^\infty(\mathbb{X}, r)$.

Remark 5. We point out that the fixed theorems in section 3.1 and section 3.2 for discrete Volterra operators are based on the finite dimensional space $\mathbb{E}^d$.

In the case that $\mathbb{X}$ is a infinite dimensional space, we can not generalize Theorem 2 and the related results to an abstract discrete Volterra operator V on $\mathcal{S}^\infty(\mathbb{X})$ by using the same technique. Our method of analysis relies on an intrinsic property of $\mathbb{E}^d$ that every bounded sequence of $\mathbb{E}^d$ has a convergent subsequence.

4 Existence theorems for discrete Volterra equations

Applying fixed point theorems established in the previous sections to certain discrete Volterra equations, we obtain several corresponding existence theorems of solutions.

4.1 Global Existence Theory

Theorem 10. *Let $V : \boldsymbol{\ell}^\infty(\mathbb{E}^d) \to \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ be a Volterra operator. Suppose that V_m $(m \geq 0)$, the restriction of V on $\mathcal{S}_m^\infty(\mathbb{E}^d)$ defined by (16), are continuous on $\mathcal{S}_m^\infty(\mathbb{E}^d)$. Then, the Volterra equation*

$$x = Vx \quad (x \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)) \tag{28}$$

has a solution $\phi \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ if and only if the equation

$$x = V_m x \quad (x \in \mathcal{S}_m^\infty(\mathbb{E}^d)) \tag{29}$$

has a solution $\phi_m \in \mathcal{S}_m^\infty(\mathbb{E}^d)$ for each $m \geq 0$, and there exists $M > 0$ such that $|\phi_m|_m^\infty \leq M$ for all $m \geq 0$.

Proof. Suppose that (28) has a solution $\phi = \{\phi(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$. Let $M = \|\phi\|_\infty$ and $\phi_m = \{\phi(n)\}_{0\leq n\leq m}$, $m = 0, 1, \ldots$. It is obvious that ϕ_m is a solution of (29) and $|\phi_m|_m^\infty \leq M$ for each $m \geq 0$. Conversely, suppose (29) has a solution $\phi_m \in \mathcal{S}_m^\infty(\mathbb{E}^d)$ for each $m \geq 0$, and there exists $M > 0$ such that $|\phi_m|_m^\infty \leq M$ for all $m \geq 0$. Using the same technique as that in the proof of Theorem 2, one can construct a solution $\phi = \{\phi(n)\}_{n\geq 0}$ of (28), which is in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$ and $\|\phi\|_\infty \leq M$. $\square$

Let us consider some examples of discrete Volterra equations. The first one is

$$x(n) = f(n) + \lambda \sum_{j=0}^{n} g(n, j, x(j)), \quad n \geq 0, \tag{30}$$

where $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{E}^d \to \mathbb{E}^d$ is a mapping, $f(n) \in \mathbb{E}^d$ $(n \geq 0)$ are given vectors, λ a real parameter, and $x(n) \in \mathbb{E}^d$ $(n \geq 0)$ are unknown vectors to be determined.

Corollary 4. *Suppose that $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{E}^d \to \mathbb{E}^d$ in (30) is continuous for $0 \leq j \leq n$, $n \geq 0$. If there exists a number $M > 0$ such that*

$$\sup_{n \geq 0} \sum_{j=0}^{n} |g(n, j, x)| \leq M \quad \textit{for all} \quad x \in \mathbb{E}^d,$$

then for any $\{f(n)\}_{n \geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$, the equation (30) has at least one bounded solution for any finite λ.

Proof. Define a Volterra operator V on $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$ as follows

$$(V\phi)(n) = f(n) + \lambda \sum_{j=0}^{n} g(n, j, \phi(j)), \quad n \geq 0, \quad \phi = \{\phi(n)\}_{n \geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d).$$

Then $|(V\phi)(n)| \leq |f(n)| + |\lambda| \sum_{j=0}^{n} |g(n, j, \phi(j))| \leq \|\{f(n)\}\|_\infty + |\lambda| M$, or $\|V\phi\|_\infty \leq \|\{f(n)\}\|_\infty + |\lambda| M$ for any $\phi = \{\phi(n)\}_{n \geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$. Thus V is a bounded Volterra operator on $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$. It is obvious that each coordinate mapping v_n of V given by

$$v_n(\phi) = f(n) + \lambda \sum_{j=0}^{n} g(n, j, \phi(j)), \quad \phi = \{\phi(m)\}_{0 \leq m \leq n} \in \mathcal{S}_n^\infty(\mathbb{E}^d),$$

is continuous mapping on $\mathcal{S}_n^\infty(\mathbb{E})$ $(n \geq 0)$. It follows from Theorem 2 that V has at least one fixed point in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$, which solves (30). □

With the same technique, we can prove corresponding results for more complicated discrete Volterra equations such as (2), (3) and (15). For example, we here give one result about (2).

Corollary 5. *Suppose that* $g_1 : \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{E}^d \to \mathbb{E}^d$ *and* $g_2 : \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{E}^d \times \mathbb{E}^d \to \mathbb{E}^d$ *are continuous mappings for each pair of integers. If there exists two numbers* $M_i > 0$, $i = 1, 2$, *such that*

$$\sup_{n\geq 0} \sum_{j=0}^{n} |g_1(n, j, x)| \leq M_1 \quad \textit{for all} \quad x \in \mathbb{E}^d$$

and

$$\sup_{n\geq 0} \sum_{j=0}^{n} \sum_{k=0}^{n} |g_2(n, j, k, x, y)| \leq M_2 \quad \textit{for all} \quad x, y \in \mathbb{E}^d,$$

then the equation (2), namely,

$$x(n) = f(n) + \sum_{j=0}^{n} g_1(n, j, x(j)) + \sum_{k=0}^{n} \sum_{j=0}^{n} g_2(n, j, k, x(j), x(k)) \quad (n \geq 0)$$

has at least one bounded solution.

Suppose that the mapping g in (30) satisfies the following hypotheses:

(**H1**) $g(n, j, x)$ is continuous in $x \in \mathbb{E}^d$ for each pair $(n, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $g(n, j, x) = 0$ if $j > n$.

(**H2**) For each bounded subset Ω of $\mathbb{E}^d$ there exist a number $M > 0$ and a mapping $Q_v : \mathbb{Z}^+ \times \mathbb{Z}^+ \times \Omega \to [0, \infty)$ such that $|g(n, j, x)| \leq Q_v(n, j)$ $(0 \leq j \leq n,\ x \in \Omega)$ and $\sup_{n\geq 0} \sum_{j=0}^{n} Q_v(n, j) = M < \infty$.

Under these conditions, we have the following theorem.

Theorem 11. *Suppose that hypotheses (H1)-(H2) are satisfied. Then for any given* $\{f(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$, *there exists a number* $M_f > 0$ *such that equation (30) has at least one bounded solution* $\{x(n)\}_{n\geq 0}$, *namely,* $\{x(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ *for all* λ *satisfying* $0 \leq |\lambda| M_f \leq 1$.

Proof. For fixed $f = \{f(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$, define a set $\Omega \in \mathbb{E}^d$ as

$$\Omega = \{x \ |x \in \mathbb{E} \quad \text{and there exists } n \ (n \geq 0) \text{ such that } |x - f(n)| \leq 1\}.$$

Since $f \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$, Ω is a bounded subset of $\mathbb{E}^d$. By (H2), there exists a number $M_f > 0$ and a mapping $Q_v : \mathbb{Z}^+ \times \mathbb{Z}^+ \times \Omega \to [0, \infty)$ such that

$$|g(n, j, x)| \leq Q_v(n, j) \quad (0 \leq j \leq n, \quad x \in \Omega) \tag{31}$$

and

$$\sup_{n\geq 0}\sum_{j=0}^{n} Q_v(n,j) = M_f < \infty. \tag{32}$$

For any $\phi = \{\phi(n)\}_{n\geq 0} \in B^\infty(f,1)$, we define an operator V on $B^\infty(f,1)$ as

$$(V\phi)(n) = f(n) + \lambda \sum_{j=0}^{n} g(n,j,\phi(j)), \quad n \geq 0. \tag{33}$$

If we define a mapping $v_i : B_i^\infty(f\rceil_i, 1) \to \mathbb{E}^d$, $i \geq 0$, by the relation

$$v_i(\phi(0),\ldots,\phi(i)) = f(i) + \lambda \sum_{j=0}^{i} g(i,j,\phi(j)) \quad \text{for any } \{\phi(j)\}_{0\leq j\leq i} \in \mathcal{S}_i^\infty(\mathbb{E}),$$

then v_i $(i \geq 0)$ are coordinate mappings of V and V is a discrete Volterra operator. For each $m \geq 0$, define a mapping $V_m : B_m^\infty(f\rceil_m, 1) \to \mathcal{S}_m^\infty(\mathbb{E}^d)$ as follows:

$$(V_m\phi)(n) = v_n(\phi(0),\ldots,\phi(n)) \quad \text{for} \quad \phi = \{\phi(n)\}_{n=0}^{m} \in B_m^\infty(f\rceil_m, 1).$$

Notice that if $\{\phi(n)\}_{n=0}^{m} \in B_m^\infty(f\rceil_m, 1)$, then $\phi(n) \in \Omega$ for each n, $0 \leq n \leq m$. It follows from (31) and (32) that

$$|(V_m\phi)(n) - f(n)| \leq |\lambda| \sum_{j=0}^{n} |g(n,j,\phi(j))| \leq |\lambda| \sum_{j=0}^{n} Q_v(n,j) \leq |\lambda| M_f \leq 1.$$

Thus, $|(V_m\phi) - f\rceil_m|_m^\infty \leq 1$, which implies that the mapping V_m maps $B_m^\infty(f\rceil_m, 1)$ to itself. Notice $\mathcal{S}_m^\infty(\mathbb{E}^d)$ is a finite dimensional space and $B_m^\infty(f\rceil_m, 1)$ is a convex, bounded subset of $\mathcal{S}_m^\infty(\mathbb{E}^d)$, V_m has one fixed point $\phi_m \in B_m^\infty(f\rceil_m, 1)$ and $|\phi|_m^\infty \leq \|f\|_\infty + 1$ by Proposition 5 (a). It follows from Theorem 3 that V has at least one fixed point in $B^\infty(f,1)$, equivalently, (30) has at least one bounded solution. $\square$

Remark 6. If for some bounded subset in $\mathbb{E}^d$, we can chose a constant M in (H2) such that $M \leq 1$, then (30) has at least one bounded solution for $\lambda = 1$.

One frequent special form of (30) is

$$x(n) = f(n) + \sum_{j=0}^{n} K(n,j) g(j, x(j)), \quad n \geq 0, \tag{34}$$

where $\{K(n,m)\}_{0\le m\le n}$, $n\ge 0$, are $d\times d$ matrices, $g:\mathbb{Z}^+\times\mathbb{E}^d\to\mathbb{E}^d$ is a mapping.

We first show that if g in (34) is bounded, then (34) has at least one solution.

Corollary 6. *Suppose that $g:\mathbb{Z}^+\times\mathbb{E}^d\to\mathbb{E}^d$ in (34) is continuous mapping on $\mathbb{E}^d$ for each $j\in\mathbb{Z}^+$ and there exists a number $M>0$ such that*

$$\sup_{n\ge 0}\sum_{j=0}^{n}|K(n,j)|\le M. \tag{35}$$

If there exists a number $M_1>0$ such that $|g(j,x)|\le M_1$ for all $j\ge 0$ and $x\in\mathbb{E}^d$, then for any $\{f(n)\}_{n\ge 0}\in\boldsymbol{\ell}^\infty(\mathbb{E}^d)$, the equation (34) has at least one solution.

Proof. The proof is very similar to that of Corollary 4 and is omitted. □

For unbounded g in (34), we make the following assumption

(**H3**) For any $x,y\in\mathbb{E}^d$ and all $j\ge 0$, g satisfies $|g(j,x)-g(j,y)|\le\lambda|x-y|$ for sufficiently small $\lambda>0$.

Theorem 12. *Let (H3) and (35) hold. Then for any fixed $f=\{f(n)\}_{n\ge 0}\in\boldsymbol{\ell}^\infty(\mathbb{E}^d)$, the equation (34) has one unique bounded solution for small λ in (H3).*

Proof. For any $\phi=\{\phi(n)\}_{n\ge 0}$, define an operator V on $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$ by the relation $(V\phi)(n)=f(n)+\sum_{j=0}^{n}K(n,j)g(j,\phi(j)),\quad n\ge 0$. If $\psi\in\boldsymbol{\ell}^\infty(\mathbb{E}^d)$, it follows that

$$\begin{aligned}|(V\phi)(n)-(V\psi)(n)| &\le \sum_{j=0}^{n}|K(n,j)||g(j,\phi(j))-g(j,\psi(j))|\\ &\le \sum_{j=0}^{n}|K(n,j)|\lambda|\phi(j)-\psi(j)|\le\lambda M\|\phi-\psi\|_\infty.\end{aligned}$$

Thus, $\|(V\phi)-(V\psi)\|_\infty\le\lambda M\|\phi-\psi\|_\infty$. If λ satisfies $0<\lambda<1/M$, then V is a contraction operator on $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$. Thus, V has a unique fixed point in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$ by Theorem 8. This completes the proof. □

The assumption (H3) guarantees that equation (34) has a unique bounded solution. If we replace (H3) with a more general condition (H4):

(**H4**) For any $x \in \mathbb{E}^d$ and all $j \geq 0$, $|g(j,x)| \leq \lambda |x|$ for sufficiently small $\lambda > 0$,

we conclude that equation (34) has at least a bounded solution.

Theorem 13. *Let (H4) and (35) hold. Then for fixed $f = \{f(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$, the equation (34) has at least one bounded solution for small λ in (H4).*

Proof. For each $m \geq 0$, define V_m on $B^\infty_m(f\rceil_m, 1) \subset \mathcal{S}^\infty_m(\mathbb{E})$ by

$$(V_m\phi)(n) = f(n) + \sum_{j=0}^{n} K(n,j)g(j,\phi(j)), \quad 0 \leq n \leq m,$$

for any $\phi = \{\phi(n)\}_{0\leq n\leq m} \in B^\infty_m(f\rceil_m, 1)$. We have

$$\begin{aligned} |(V_m\phi)(n) - f(n)| &\leq \sum_{j=0}^{n} |K(n,j)||g(j,\phi(j))| \\ &\leq \lambda \sum_{j=0}^{n} |K(n,j)||\phi(j)| \\ &\leq \lambda M |\phi|^\infty_m \leq \lambda M(1 + \|f\|_\infty). \end{aligned}$$

If $\lambda \leq 1/M(1 + \|f\|_\infty)$, then V_m maps $B^\infty_m(f\rceil_m, 1)$ to $B^\infty_m(f\rceil_m, 1)$. Since $B^\infty_m(f\rceil_m, 1)$ is a closed, bounded, convex subset of the finite dimensional Banach space $\mathcal{S}^\infty_m(\mathbb{E}^d)$, V_m has at least one fixed point $\phi_m \in B^\infty_m(f\rceil_m, 1)$ and $|\phi_m|^\infty_m \leq (1 + \|f\|_\infty)$ for each $m \geq 0$ by Proposition 5. It follows from Theorem 3 that V has at least one fixed point in $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$. This completes the proof. □

4.2 Local Existence Theory

The assumption (H3) requires the mapping g in (34) satisfy the global Lipschitz condition. But in some cases, we need to consider (34), namely

$$x(n) = f(n) + \sum_{j=0}^{n} K(n,j)g(j,x(j)), \quad n \geq 0, \tag{36}$$

under the following more general condition:

(**H5**) $g(j,0) \equiv 0$ for all $j \geq 0$ and for any $\tau > 0$, there exists a number $\sigma > 0$ such that

$$|g(j,x)| \leq \tau |x| \quad \text{for all} \quad j \geq 0 \quad \text{when} \quad |x| \leq \sigma. \tag{37}$$

Before the main result, we need the following Lemmas.

Lemma 1. *Suppose, for some positive numbers r and s, $g(j,\cdot)$ maps $B_0^\infty(r)$ to $B_0^\infty(s)$ continuously for $0 \leq j \leq m$. If $\psi \in \mathcal{S}_m^\infty(\mathbb{E}^d)$ and satisfies*

$$|\psi|_m^\infty + Ms \leq r, \tag{38}$$

then the equation

$$x(n) = \psi(n) + \sum_{j=0}^{n} K(n,j) G_j(x(j)), \quad 0 \leq n \leq m, \tag{39}$$

has at least one solution $\phi_m = \{\phi_m(n)\}_{n=0}^m \in \mathcal{S}_m^\infty(\mathbb{E}^d)$ with $|\phi_m|_m^\infty \leq r$.

Proof. For any $\phi = \{\phi(n)\}_{n=0}^m \in B_m^\infty(\mathbb{E}^d, r)$, the ball of radius $r > 0$ in $\mathcal{S}_m^\infty(\mathbb{E}^d)$, and fixed $\psi \in \mathcal{S}_m^\infty(\mathbb{E}^d)$ satisfying (38), define the operator V_m by the relation

$$(V_m\phi)(n) = \psi(n) + \sum_{j=0}^{n} K(n,j) g(j,\phi(j)), \quad 0 \leq n \leq m.$$

Notice that if $\phi = \{\phi(n)\}_{n=0}^m \in B_m^\infty(\mathbb{E}^d, r)$, then $\phi(n) \in B_0^\infty(r)$ for $0 \leq n \leq m$. It follows from the assumptions of Lemma 1 and (38) that

$$\begin{aligned} |V_m(\phi)(n)| &\leq |\psi(n)| + \sum_{j=0}^{n} |K(n,j)||g(j,\phi(j))| \\ &\leq |\psi|_m^\infty + Ms \leq r. \end{aligned}$$

Thus V_m maps $B_m^\infty(\mathbb{E}^d, r)$ to $B_m^\infty(\mathbb{E}^d, r)$ and is continuous. Since $B_m^\infty(\mathbb{E}^d, r)$ is a closed, bounded, convex subset of the finite dimensional Banach space $\mathcal{S}_m^\infty(\mathbb{E}^d)$, it follows from Proposition 5 that V_m has at least one fixed point in $B_m^\infty(\mathbb{E}^d, r)$. □

Lemma 2. *Suppose that (35) holds and that for some positive number r and s, $g(j,\cdot)$ maps $B_0^\infty(r)$ to $B_0^\infty(s)$ continuously, for all $j \geq 0$. If $f = \{f(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ and satisfies*

$$|f|_\infty + Ms \leq r, \tag{40}$$

then the equation (36) has at least one solution $x = \{x(n)\}_{n\geq 0} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ with $\|\{x(n)\}\|_\infty \leq r$.

Proof. By virtue of Theorem 3 and Lemma 1, the proof is obvious. □

We are now in a position to present our main result.

Theorem 14. *Suppose that (35) and (H5) hold. Then for each λ, $0 < \lambda < 1$, there exists $\varepsilon_0 > 0$ such that $0 < \varepsilon < \varepsilon_0$ and $\|\{f(n)\}\|_\infty \leq \lambda\varepsilon$ imply that there exists at least one solution $\{x(n)\}_{n\geq 0}$ of (36) satisfying $\|\{x(n)\}\|_\infty \leq \varepsilon$. If $g(j,x)$ $(j \geq 0)$ is locally Lipschitz, there is only one such $\{x(n)\}_{n\geq 0}$.*

Proof. Given $0 < \lambda < 1$, choose $\varepsilon_0 > 0$ such that

$$|g(j,x)| \leq \frac{1-\lambda}{M}|x| \qquad \text{for } j \geq 0 \text{ and } |x| \leq \varepsilon_0,$$

where M is defined in (35). Let $0 < \varepsilon < \varepsilon_0$ and $\|\{f(n)\}\|_\infty \leq \lambda\varepsilon$. For $\phi = \{\phi(n)\} \in \boldsymbol{\ell}^\infty(\mathbb{E}^d)$ with $\|\phi\|_\infty \leq \varepsilon$, we have

$$\|f\|_\infty + M\frac{1-\lambda}{M}\varepsilon \leq \varepsilon.$$

On the other hand, $g(j,\cdot)$ maps $B_0^\infty(\varepsilon)$ to $B_0^\infty(\frac{1-\lambda}{M}\varepsilon)$ continuously, for each $j \geq 0$. It follows from Lemma 2 that (36) has at least one solution $\{x(n)\}_{n\geq 0}$ for given $\{f(n)\}_{n\geq 0}$ with $\|\{x(n)\}\|_\infty \leq \varepsilon$. □

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